UNIVERSITY OF GRONINGEN
Faculty of Science and Engineering

WBIE048-05 Control Engineering

Exam 2020

Date: Friday, October 30th, 2020, Time: 16:00-18:00

Lecturer and Coordinator

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- Please write completely your name, student ID number and study program.
- You can choose to answer FOUR questions out of FIVE available questions. If you answer all FIVE of them, extra points can be granted (if correct).
- For every question, write your answer neatly on blank papers and write down your name and student number on the top of each page.
- This is an OPEN book exam and you are allowed to use calculator or computer. **Use of WIFI is forbidden.**
- **Please write down your answer clearly and with proper argumentation whenever needed. Providing only the final answers without proper argumentation is NOT acceptable and will NOT be graded.** Please, write your answer using a pen, not a pencil.
- **Whenever we think is appropriate, a follow-up ORAL examination to suspected cases will be arranged before the final grade is determined. In this case, the follow-up oral examination will be based on the questions of the final exam and the final grade will be based on the same weighting factor as before where the adjusted grade from the oral examination will be used instead that replaces the final exam grade.**
- If you return the sheets, then your exam will be graded, unless you explicitly write ”do not grade” on the first page. If your exam is graded, then the grade will be registered.
- Hints are sometimes providied after a questions to be used *only in the case* in which you did not answer a previous question.

For the grader only

	Exercise 1	**Exercise 2**	**Exercise 3**	**Exercise 4**	**Exercise 5**
Points					X
Bonus	X	X	X	X	

1. **Exercise 1 (10pts).** Consider the air-levitation system shown in Figure 1. This system has a fan in the bottom, which "injects" air in-flux that acts on a air-balloon of mass m in a closed-cylinder.

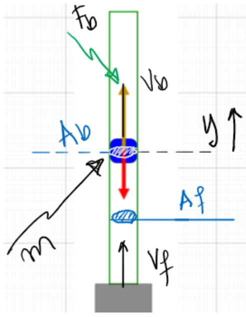

Figure 1: Air levitation system.

To simplify the modeling (control-oriented), assume that the motion of the balloon occurs only in the y direction as shown in Figure 1. Since the cylinder is closed, the air flux inside is constant. Then, the following *volumetric* flow rate relation holds

$$A_f V_f = A_b V_b, \tag{1}$$

where A_f and A_b represent the transversal area of the cylinder and the balloon, respectively; and V_f and V_b are the air velocity on the transversal sections of the fan and balloon, respectively. By the Bernoulli principle, the drag force F_b acting on the balloon is

$$F_b = \frac{1}{2} A_b \rho V_b^2$$

with ρ the flux density. Due the drag force, there exists a corresponding viscous friction force (opposite to F_b) given by

$$F_{\text{fric}} = \frac{1}{2} C \rho A_b \dot{y} \tag{2}$$

where C is the drag coefficient for the balloon.

(a) (2pts) Find a control-oriented mathematical model for the system in Figure 1.

SOLUTION: From the free body diagram in Figure 1 and Newton's second law, we have

$$\begin{aligned} m\ddot{y} &= F_b - mg - F_{\text{fric}} \\ \Rightarrow m\ddot{y} &= \frac{1}{2} A_b \rho V_b^2 - mg - \frac{1}{2} C \rho A_b \dot{y}. \end{aligned} \tag{3}$$

From (1) we know that

$$V_b = \frac{A_f V_f}{A_b}. \tag{4}$$

Then,

$$m\ddot{y} = \frac{1}{2} \rho \frac{A_f^2 V_f^2}{A_b} - mg - \frac{1}{2} C \rho A_b \dot{y}. \tag{5}$$

Since the drag force is a function of the fan's velocity V_f, then the control input is $u = V_f$, and the control oriented model is given by

$$m\ddot{y} + \frac{1}{2} C \rho A_b \dot{y} + mg = \frac{1}{2} \rho \frac{A_f^2}{A_b} u^2. \tag{6}$$

Note that this model is linear in the ball's position y, but nonlinear in the input u.

(b) (2pts) Propose a state-space representation for the differential equation of point (a).

SOLUTION:
Let $x_1 = y$ and $x_2 = \dot{y}$ the state variables. Then,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}\left[-mg - \frac{1}{2}C\rho A_b x_2 + \frac{1}{2}\rho\frac{A_f^2}{A_b}u^2\right],\end{aligned} \tag{7}$$

or in vector form

$$\underbrace{\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} x_2 \\ -g - \frac{1}{2m}C\rho A_b x_2 + \frac{1}{2}\rho\frac{A_f^2}{A_b m}u^2\end{bmatrix}}_{f(x,u)} = \begin{bmatrix} f_1(x,u) \\ f_2(x,u)\end{bmatrix}. \tag{8}$$

(c) (3pts) Given a constant position for the balloon $y = Y$, determine an operation point $(\overline{x}, \overline{u})$ for the state space model.

SOLUTION:
Since the state model in (8) does not depend on the balloon's position y, explicitly, we can pick any arbitrarily constant position $\overline{y} = Y$ with respect to the fan; that is, $Y > 0$. To compute the constant operation point $(\overline{x}, \overline{u})$ we need to solve the following set of algebraic equations: $\dot{\overline{x}} = f(\overline{x}, \overline{u}) = 0$, or

$$\underbrace{\begin{bmatrix} \overline{x}_2 \\ -g - \frac{1}{2m}C\rho A_b \overline{x}_2 + \frac{1}{2}\rho\frac{A_f^2}{A_b m}\overline{u}^2\end{bmatrix}}_{f(\overline{x},\overline{u})} = \begin{bmatrix}0 \\ 0\end{bmatrix} \tag{9}$$

it follows that $\overline{x}_2 = 0$. Substitution of this in the second equation, and solving for u results in

$$-g + \frac{1}{2}\rho\frac{A_f^2}{A_b m}\overline{u}^2 = 0 \iff \overline{u} = \pm\sqrt{\frac{2A_b mg}{\rho A_f^2}}. \tag{10}$$

We take only the positive solution of u because we are interested only in wind flow in the positive direction. Hence, the operation point is

$$(\overline{x}, \overline{u}) = \left(\begin{bmatrix} Y \\ 0\end{bmatrix}, \sqrt{\frac{2A_b mg}{\rho A_f^2}}\right). \tag{11}$$

(d) (3pts) Analyze the stability of the operation point $(\overline{x}, \overline{u})$ via the linearization.

SOLUTION:
The linearized model around $(\overline{x}, \overline{u})$ is given by

$$\delta\dot{x} = A\delta x + B\delta u \tag{12}$$

with $\delta x = x - \overline{x}$ and $\delta u = u - \overline{u}$

$$A = \frac{\partial f}{\partial x}(x,u)\Big|_{(x,u)=(\overline{x},\overline{u})} = \begin{bmatrix}\frac{\partial f_1}{\partial x_1}(x,u) & \frac{\partial f_1}{\partial x_2}(x,u) \\ \frac{\partial f_2}{\partial x_1}(x,u) & \frac{\partial f_2}{\partial x_2}(x,u)\end{bmatrix}\Bigg|_{(x,u)=(\overline{x},\overline{u})}, \tag{13}$$

and

$$B = \frac{\partial f}{\partial u}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial u}(x,u) \\ \frac{\partial f_2}{\partial u}(x,u) \end{bmatrix}\bigg|_{(x,u)=(\overline{x},\overline{u})}. \tag{14}$$

The partial derivatives for A are computed as follows

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= 0, \\
\frac{\partial f_1}{\partial x_2}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= 1, \\
\frac{\partial f_2}{\partial x_1}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= 0, \\
\frac{\partial f_2}{\partial x_2}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= \frac{\partial}{\partial x_2}\left(-g - \frac{1}{2m}C\rho A_b x_2 + \frac{1}{2}\rho\frac{A_f^2}{A_b m}u^2\right)\bigg|_{(x,u)=(\overline{x},\overline{u})} \\
&= \frac{\partial}{\partial x_2}\left(-\frac{1}{2m}C\rho A_b x_2\right)\bigg|_{(x,u)=(\overline{x},\overline{u})} = -\frac{1}{2m}C\rho A_b.
\end{aligned} \tag{15}$$

Similarly, the partial derivatives for B are

$$\begin{aligned}
\frac{\partial f_1}{\partial u}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= 0, \\
\frac{\partial f_2}{\partial u}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= \frac{\partial}{\partial u}\left(-g - \frac{1}{2m}C\rho A_b x_2 + \frac{1}{2}\rho\frac{A_f^2}{A_b m}u^2\right)\bigg|_{(x,u)=(\overline{x},\overline{u})} \\
&= \frac{\partial}{\partial u}\left(\frac{1}{2}\rho\frac{A_f^2}{A_b m}u^2\right)\bigg|_{(x,u)=(\overline{x},\overline{u})} = \frac{\rho A_f^2}{A_b m}u\bigg|_{(x,u)=(\overline{x},\overline{u})}, \\
&= \frac{\rho A_f^2}{A_b m}\underbrace{\sqrt{\frac{2A_b m g}{\rho A_f^2}}}_{\overline{u}} = A_f\sqrt{\frac{2\rho g}{m A_b}}.
\end{aligned} \tag{16}$$

Substitution in (12) yields

$$\delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2m}C\rho A_b \end{bmatrix}}_{A}\delta x + \underbrace{\begin{bmatrix} 0 \\ A_f\sqrt{\frac{2\rho g}{m A_b}} \end{bmatrix}}_{B}\delta u. \tag{17}$$

Now, let us compute the eigenvalues of the linearized system matrix

$$p(s) = \det(sI - A) = \det\begin{bmatrix} s & -1 \\ 0 & s + \frac{1}{2m}C\rho \end{bmatrix} = s\left(s + \frac{1}{2m}C\rho\right) = 0, \tag{18}$$

that is, the eigenvalues of the linearization are $\{0, -\frac{1}{2m}C\rho\}$. Since there is an eigenvalue at $s = 0$ the first Lyapunov criterion can not give a conclusion about the stability of $\overline{x}$ for the nonlinear system in (8).

Hint: If you did not find the mathematical model in (a), use $m\ddot{y} = -0.5g - 2y - 0.5\dot{y} + \frac{cu^2}{4y}$.

(a) (2pts) Solution using the model given in the hint: $m\ddot{y} = -0.5g - 2y - 0.5\dot{y} + \frac{cu^2}{4y}$

(b) (2pts) Propose a state-space representation for the differential equation of point (a).

SOLUTION:
Let $x_1 = y$ and $x_2 = \dot{y}$ the state variables. Then,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}\left[-0.5g - 2x_1 - 0.5x_2 + \frac{cu^2}{4x_1}\right], \end{aligned} \tag{19}$$

or in vector form

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} x_2 \\ -\frac{0.5g}{m} - \frac{2}{m}x_1 - \frac{0.5}{m}x_2 + \frac{cu^2}{4mx_1} \end{bmatrix}}_{f(x,u)} = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix}. \tag{20}$$

(c) (3pts) Given a constant position for the balloon $y = Y$, determine an operation point $(\overline{x}, \overline{u})$ for the state space model.

SOLUTION:
To compute the constant operation point $(\overline{x}, \overline{u})$ we need to solve the following set of algebraic equations: $\dot{\overline{x}} = f(\overline{x}, \overline{u}) = 0$, or

$$\underbrace{\begin{bmatrix} \overline{x}_2 \\ -\frac{0.5g}{m} - \frac{2}{m}\overline{x}_1 - \frac{0.5}{m}\overline{x}_2 + \frac{c\overline{u}^2}{4m\overline{x}_1} \end{bmatrix}}_{f(\overline{x},\overline{u})} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21}$$

it follows that $\overline{x}_2 = 0$. Substitution of this in the second equation, and solving for u results in

$$-\frac{0.5g}{m} - \frac{2}{m}\overline{x}_1 - \frac{0.5}{m}\overline{x}_2 + \frac{cu^2}{4mx_1} = 0 \iff \overline{u} = \pm\sqrt{\frac{4\overline{x}_1}{c}\left(0.5g + 2\overline{x}_1\right)} \quad \text{with} \quad \overline{x}_1 \neq 0. \tag{22}$$

Note that the nominal controller $\overline{u}$ depends of $\overline{x}_1$, and there is no any restriction on it apart of $\overline{x}_1 \neq 0$; thus, we can take any arbitrary constant $\overline{x}_1 = Y$, with $Y \neq 0$. In this case, let us take the positive solution[1] of u. Hence, the operation point is

$$(\overline{x}, \overline{u}) = \left(\begin{bmatrix} Y \\ 0 \end{bmatrix}, \sqrt{\frac{4\overline{x}_1}{c}\left(0.5g + 2\overline{x}_1\right)}\right) \quad \text{with} \quad \overline{x}_1 \neq 0.$$

(d) (3pts) Analyze the stability of the operation point $(\overline{x}, \overline{u})$ via the linearization.

SOLUTION:
The linearized model around $(\overline{x}, \overline{u})$ is given by

$$\delta\dot{x} = A\delta x + B\delta u \tag{23}$$

with $\delta x = x - \overline{x}$ and $\delta u = u - \overline{u}$

$$A = \frac{\partial f}{\partial x}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x,u) & \frac{\partial f_1}{\partial x_2}(x,u) \\ \frac{\partial f_2}{\partial x_1}(x,u) & \frac{\partial f_2}{\partial x_2}(x,u) \end{bmatrix}\bigg|_{(x,u)=(\overline{x},\overline{u})}, \tag{24}$$

[1]Since we do not know the nature of the mathematical model, the negative solution of $\overline{u}$ is possible.

and

$$B = \frac{\partial f}{\partial u}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial u}(x,u) \\ \frac{\partial f_2}{\partial u}(x,u) \end{bmatrix}\bigg|_{(x,u)=(\overline{x},\overline{u})}. \tag{25}$$

The partial derivatives for A are computed as follows

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= 0,\\
\frac{\partial f_1}{\partial x_2}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= 1,\\
\frac{\partial f_2}{\partial x_1}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= \frac{\partial}{\partial x_1}\left(-\frac{0.5g}{m} - \frac{2}{m}x_1 - \frac{0.5}{m}x_2 + \frac{cu^2}{4mx_1}\right)\bigg|_{(x,u)=(\overline{x},\overline{u})}\\
&= -\left(\frac{2}{m} + \frac{cu^2}{4mx_1^2}\right)\bigg|_{(x,u)=(\overline{x},\overline{u})} = -\frac{2}{m} - \frac{1}{2m}\sqrt{\frac{c}{Y}(0.5g+2Y)}, \quad Y \neq 0,\\
\frac{\partial f_2}{\partial x_2}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= \frac{\partial}{\partial x_2}\left(-\frac{0.5g}{m} - \frac{2}{m}x_1 - \frac{0.5}{m}x_2 + \frac{cu^2}{4mx_1}\right)\bigg|_{(x,u)=(\overline{x},\overline{u})}\\
&= -\frac{0.5}{m}
\end{aligned} \tag{26}$$

Similarly, the partial derivatives for B are

$$\begin{aligned}
\frac{\partial f_1}{\partial u}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= 0,\\
\frac{\partial f_2}{\partial u}(x,u)\bigg|_{(x,u)=(\overline{x},\overline{u})} &= \frac{\partial}{\partial u}\left(-\frac{0.5g}{m} - \frac{2}{m}x_1 - \frac{0.5}{m}x_2 + \frac{cu^2}{4mx_1}\right)\bigg|_{(x,u)=(\overline{x},\overline{u})}\\
&= \frac{cu}{2mx_1}\bigg|_{(x,u)=(\overline{x},\overline{u})} = \sqrt{\frac{c}{m^2Y}(0.5g+2Y)}, \quad Y \neq 0.
\end{aligned} \tag{27}$$

Substitution in (23) yields

$$\delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\left(\frac{2}{m} + \frac{1}{2m}\sqrt{\frac{c}{Y}(0.5g+2Y)}\right) & -\frac{0.5}{m} \end{bmatrix}}_{A}\delta x + \underbrace{\begin{bmatrix} 0 \\ \sqrt{\frac{c}{m^2Y}(0.5g+2Y)} \end{bmatrix}}_{B}\delta u. \tag{28}$$

The characteristic equation of the linearized system is

$$\begin{aligned}
p(s) = \det(sI - A) &= \det\begin{bmatrix} s & -1 \\ \left(\frac{2}{m} + \frac{1}{2m}\sqrt{\frac{c}{Y}(0.5g+2Y)}\right) & s + \frac{0.5}{m} \end{bmatrix},\\
&= s^2 + \frac{0.5}{m}s + \left(\frac{2}{m} + \frac{1}{2m}\sqrt{\frac{c}{Y}(0.5g+2Y)}\right) = 0.
\end{aligned} \tag{29}$$

Using the Routh-Hurwitz criterion, consider the Routh's table

s^2	$1 > 0$	$\left(\frac{2}{m} + \frac{1}{2m}\sqrt{\frac{c}{Y}(0.5g+2Y)}\right)$
s^1	$\frac{0.5}{m} > 0$	0
s^0	$b_1 = \left(\frac{2}{m} + \frac{1}{2m}\sqrt{\frac{c}{Y}(0.5g+2Y)}\right) > 0$	

Since all the values in the first column are strictly positive, we conclude that poles are in the left half complex plane. Hence, by the first Lyapunov theorem, the operation point $(\overline{x},\overline{u})$ is asymptotically stable for the nonlinear system in (20).

Table 1: System's parameters

Parameter	g	H_f	A_f	m	A_b	c	ρ
Value	9.81 m/s^2	0.025 m	0.0113 m^2	0.0013 Kg	0.0154 m^2	0.1	1.225 Kg/m^3

2. **Exercise 2 (10pts).** Consider the linearized model of Exercise 1.(d), with the parameters given in Table 1

   (a) (3pts) Design a full state feedback controller such that the closed-loop system has a natural frequency $\omega_n = 0.25$ and a damping ratio $\zeta = 0.004$.

   SOLUTION:

   Using the numerical values in Table 1, matrices A and B are respectively given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.7255 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 12.3812 \end{bmatrix}. \tag{30}$$

   The reachability matrix is

$$W_r = [BAB] = \begin{bmatrix} 0 & 12.3812 \\ 12.3812 & -8.9835 \end{bmatrix} \Rightarrow \text{rank}(W_r) = 2. \tag{31}$$

   This means that the pair (A, B) is reachable, and we can construct the feedback law $\delta u = -K\delta x + k_r r$. The desired poles are the roots of the target characteristic polynomial

$$p_{\text{tg}}(s) = s^2 2\zeta\omega_n s + \omega_n^2 = s^2 + 0.002s + 0.0625 = s^2 + \alpha_1 s + \alpha_2, \tag{32}$$

   which are $\{-0.001 \pm j0.2499\}$. To propose the canonical reachable form, consider the characteristic polynomial associated to A in (18) with the numerical values in Table (1), that is,

$$p(s) = s^2 + 0.72565s = s^2 + a_1 s + a_2. \tag{33}$$

   Then, the reachable canonical form is given by

$$\dot{z} = \underbrace{\begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}}_{\tilde{A}} z + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\tilde{B}} u = \begin{bmatrix} -0.7256 & 0 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u. \tag{34}$$

   The reachability matrix $\tilde{W}_r = [\tilde{B}|\tilde{A}\tilde{B}]$ is

$$\tilde{W}_r = \begin{bmatrix} 1 & -0.7256 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(\tilde{W}_r) = 2. \tag{35}$$

   The feedback gain is

$$K = [\alpha_1 - a_1 \;\; \alpha_2 - a_2]\tilde{W}_r W_r^{-1} = [0.005047 \;\; -0.058441]. \tag{36}$$

   Here the reference $r = 0$ because we want $\delta x = x - \overline{x} \to 0$, which is equivalent to say that $x \to \overline{x}$ (locally) for the nonlinear system. However, let us compute the gain k_r, according to the formula

$$k_r = -\frac{1}{C(A - BK)^{-1}B} = -0.08076, \tag{37}$$

   where $C = [1\ 0]$, because we are interested in a constant value $y = Y$ for the balloon's position, i.e., to $y = x_1$.

(b) (2pts) Write explicitly the controller to be implemented in the nonlinear system, and verify that the operation point of Exercise 1.(c) is asymptotically stable.

SOLUTION:

From the controller for the linearized system $\delta u = -K\delta x$, we can obtain the controller u for the nonlinear system (8) using the operational point in (11), as follows:

$$u = \delta u + \overline{u} = -K(x - \overline{x}) + \overline{u} = -[0.005047 \ \ -0.058441] \begin{bmatrix} x_1 - Y \\ x_2 - 0 \end{bmatrix} + 1.5847. \quad (38)$$

It follows that the nonlinear system (8) (after substitutions of the values in Table 1) in closed-loop with the control scheme (38) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -9.81 - 0.7256x_2 + 3.9066\left(-0.005047(x_1 - Y) + 0.058441x_2 + 1.5847\right)^2 \end{bmatrix}. \quad (39)$$

Compute the closed-loop equilibrium point $\overline{x}$, which is the solution to

$$\begin{bmatrix} \overline{x}_2 \\ -9.81 - 0.7256\overline{x}_2 + 3.9066\left(-0.005047(\overline{x}_1 - Y) + 0.058441\overline{x}_2 + 1.5847\right)^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (40)$$

Then, $\overline{x}_2 = 0$ and $-9.81 + 3.9066\left(-0.0001(\overline{x}_1 - Y) + 1.5847\right)^2 = 0$, which results in $(\overline{x} = (Y, 0))$ as expected. To check whether or not this equilibrium point is stable, consider the Jacobian of the closed-loop system (39) at $x = \overline{x}$ as follows

$$A_{\text{cl}} = \begin{bmatrix} 0 & 1 \\ -0.0394 & -0.26898 \end{bmatrix}, \quad (41)$$

whose associated characteristic polynomial is

$$p_{\text{cl}}(s) = s^2 + 0.2689s + 0.0394. \quad (42)$$

Using the Routh-Hurwitz criterion we have

s^2	1	0.0394
s^1	0.2689	0
s^0	$b_1 = 0.0394$	

Therefore, the equilibrium point $\overline{x}$ is an (locally) asymptotically stable equilibrium point for the nonlinear system.

(c) (3pts) Design an observer and write it explicitly.

SOLUTION:

Recall the output equation matrix $C = [1 \ 0]$. Then, the observability matrix is

$$W_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (43)$$

which clearly is full-rank and the pair (A, C) is observable. Take as target eigenvalues for the observer to $\{-0.005 \pm j1.25\}$. Using the duality observability/reachability theorem and the Matlab command

$$Lo = \text{place}(A^\top, C^\top, -0.005 \pm j1.25)^\top,$$

the observer's gain L is given by $L = [-0.7155, \ 2.08170]^\top$. The explicit expression of the observer is therefore

$$\delta\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -0.7255 \end{bmatrix} \delta\hat{x} + \begin{bmatrix} 0 \\ 12.3812 \end{bmatrix} \delta u + \begin{bmatrix} -0.7155 \\ 2.08170 \end{bmatrix} (y - \delta\hat{y}). \quad (44)$$

I used $\delta\hat{x}$ instead of $\hat{x}$, because the observer is designed for the linearized system[2].

[2] If you fail to do this, I can understand that it is not clear for you what the Jacobian linearization is.

(d) (2pts) Design a dynamic output-feedback controller and write it explicitly.
SOLUTION:
By the separation principle theorem, the observer and feedback gains, L and K can be designed separately. Hence, items (b) and (c) solve this step. The explicit form of the dynamic output feedback is

$$\begin{aligned} \delta\dot{\hat{x}} &= (A - BK - LC)\delta\hat{x} + Bk_r r + Ly \\ \delta u &= -K\delta\hat{x} + k_r r \end{aligned} \tag{45}$$

or

$$\begin{aligned} \delta\dot{\hat{x}} &= \begin{bmatrix} 0.7155 & 1 \\ -2.1442 & -0.002 \end{bmatrix} \delta\hat{x} + \begin{bmatrix} 0 \\ 12.3812 \end{bmatrix} r + \begin{bmatrix} -0.7155 \\ 2.08170 \end{bmatrix} y \\ \delta u &= -0.00504\delta x_1 + 0.0584\delta x_2 - 0.0807r. \end{aligned} \tag{46}$$

3. **Exercise 3 (10pts).** Consider the prototype second order system

$$P(s) = \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2} \tag{47}$$

in closed-loop with the PID controller

$$C(s) = K_p + \frac{K_i}{s} + K_d s. \tag{48}$$

(a) Show that the PID controller allows arbitrary pole placement.
SOLUTION:
The loop transfer function $L(s) = P(s)C(s)$ is given by

$$\begin{aligned} L(s) &= \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2}\left(K_p + \frac{K_i}{s} + K_d s\right) \\ &= \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2}\left(\frac{K_d s^2 + K_p s + K_i}{s}\right) \\ &= \frac{\omega_n^2(K_d s^2 + K_p s + K_i)}{s^3 + 2\omega_n\zeta s^2 + \omega_n^2 s}. \end{aligned} \tag{49}$$

The closed-loop transfer function is

$$\begin{aligned} G(s) &= \frac{L(s)}{1 + L(s)} \\ &= \frac{\omega_n^2(K_d s^2 + K_p s + K_i)}{\omega_n^2(K_d s^2 + K_p s + K_i) + s^3 + 2\omega_n\zeta s^2 + \omega_n^2 s} \\ &= \frac{\omega_n^2(K_d s^2 + K_p s + K_i)}{s^3 + (2\omega_n\zeta + \omega_n^2 K_d)s^2 + (\omega_n^2 + \omega_n^2 K_p)s + \omega_n^2 K_i}. \end{aligned} \tag{50}$$

By the Routh-Hurwitz criterion, it is clear that properly choosing the PID's gains, the closed-loop poles can be arbitrarily placed in the (left or right half) complex plane.

(b) Show that the system steady-state error is zero despite disturbances, see Figure 2.
SOLUTION:
The two constant inputs in the Laplace domain are given by $R(s) = r/s$ and $V(s) =$

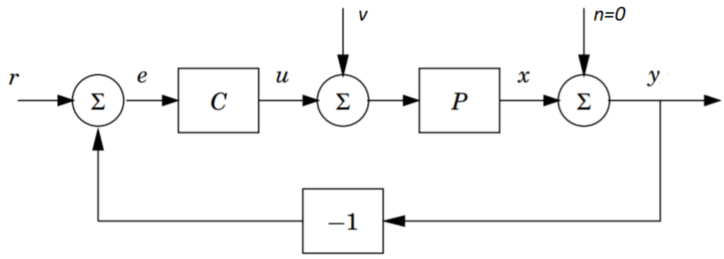

Figure 2: Feedback system with reference r, load disturbance v, and noise $n = 0$.

v/s. Then, the error is given by

$$\begin{aligned} E(s) &= \frac{1}{1+L(s)}R(s) - \frac{P(s)}{1+L(s)}V(s) \\ &= \frac{s^3 + 2\omega_n\zeta s^2 + \omega_n^2 s}{s^3 + (2\omega_n\zeta + \omega_n^2 K_d)s^2 + (\omega_n^2 + \omega_n^2 K_p)s + \omega_n^2 K_i} R(s) \\ &+ \frac{\omega_n^2 s}{s^3 + (2\omega_n\zeta + \omega_n^2 K_d)s^2 + (\omega_n^2 + \omega_n^2 K_p)s + \omega_n^2 K_i} W(s). \end{aligned} \tag{51}$$

The steady-state error e_{ss} can be computed using the final value theorem

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{s(s^3 + 2\omega_n\zeta s^2 + \omega_n^2 s)}{s^3 + (2\omega_n\zeta + \omega_n^2 K_d)s^2 + (\omega_n^2 + \omega_n^2 K_p)s + \omega_n^2 K_i} \frac{r}{s} \\ &- \lim_{s\to 0} \frac{s\omega_n^2 s}{s^3 + (2\omega_n\zeta + \omega_n^2 K_d)s^2 + (\omega_n^2 + \omega_n^2 K_p)s + \omega_n^2 K_i} \frac{v}{s} = 0 \end{aligned} \tag{52}$$

4. **Exercise 4 (10pts).** Consider the linearized model of Exercise 1.(d) with the parameter in Table 1. Design a PI controller for the system such that the system has a phase margin of $\varphi_m = 30°$ and a crossover frequency of $\omega_c = 2$ rad/sec.

SOLUTION:

Let us compute the transfer function corresponding to the linear state equation given by the pair (A,B) in (30), with output matrix $C = [1\ 0]$, according to the formula

$$\begin{aligned} P(s) &= C(sI - A)^{-1}B \\ &= [1\ 0]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -0.7255 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 12.3812 \end{bmatrix} \\ &= \frac{12.3812}{s(s+0.7255)} = \frac{12.3812}{s^2 + 0.7255s}. \end{aligned} \tag{53}$$

Consider the PI controller in the frequency domain given by

$$C(s) = k_p + k_i\frac{1}{s} = k_p\left(1 + \frac{k_i}{k_p}\frac{1}{s}\right). \tag{54}$$

First all, notice that the main goal of this problem is not to track a reference (with zero-steady state), but to ensure frequency domain specifications for the controlled system. This can be done[3] in either open-loop or closed-loop.

[3]For open-loop control, imagine that you want a building (the plant $P(s)$) that can resist earthquakes. This means that you need to add something (the controller $C(S)$) that modifies the frequency specifications (natural frequency, band-width, phase margin, gain margin) of the building. You can do it in an passive or active manner. For the first, typically springs and dampers are directly connected to the system, and no feedback may be required. For the active approach, actuators are included, which require feedback, to compare to a given reference.

For the open loop case, we consider the loop transfer function given by

$$L(s) = C(s)P(S) = 12.3812k_p\left(1+\frac{k_i}{k_p}\frac{1}{s}\right)\left(\frac{1}{s^2+0.7255s}\right), \tag{55}$$

which already contains the controller transfer function $C(s)$ acting on the plant $P(s)$. Thus, at the crossover frequency $\omega = \omega_c = 2$rad/sec, we need to find k_p and k_i such that[4]

$$M = |L(j\omega)|_{\omega=\omega_c} = 1, \qquad \text{and} \qquad \angle L(j\omega)|_{\omega=\omega_c} + 180^\circ = \phi_m. \tag{56}$$

To this end, we need first to compute $L(j\omega)$. Then

$$L(j\omega) = 12.3812k_p\left(1+\frac{k_i}{k_p}\frac{1}{j\omega}\right)\left(\frac{1}{(j\omega)^2+0.7255j\omega}\right). \tag{57}$$

The magnitude and phase of the transfer function $L(j\omega)$, respectively, are

$$\begin{aligned} |L(j\omega)|_{\omega=\omega_c} &= 12.3812k_p\sqrt{1+\frac{k_i^2}{k_p^2}\frac{1}{\omega_c^2}}\frac{1}{\sqrt{\omega_c^4+0.7255^2\omega_c^2}} \\ \angle L(j\omega)|_{\omega=\omega_c} &= \arctan\left(-\frac{k_i}{k_p}\frac{1}{\omega_c}\right) - \arctan\left(-\frac{0.7255}{\omega_c}\right) \end{aligned} \tag{58}$$

Now, let us find the gains k_p and k_i from conditions in (56). It yields the following two equations, with $\omega_c = 2$rad/sec and $\phi_m = 30^\circ$, and k_p and k_i the unknowns

$$\begin{aligned} 12.3812k_p\sqrt{1+\frac{k_i^2}{k_p^2}\frac{1}{4}}\frac{1}{\sqrt{16+2.1054}} &= 1 \\ \arctan\left(-\frac{k_i}{k_p}\frac{1}{2}\right) - \arctan(-0.36627) + 180^\circ &= 30^\circ \end{aligned} \tag{59}$$

Solving for k_p and k_p we get the following values

$$k_p = 0.3383, \qquad \text{and} \qquad k_i = -0.3548k_p = -0.12. \tag{60}$$

Just to illustrate (no necessary in the exam), let us plot the Bode diagram.

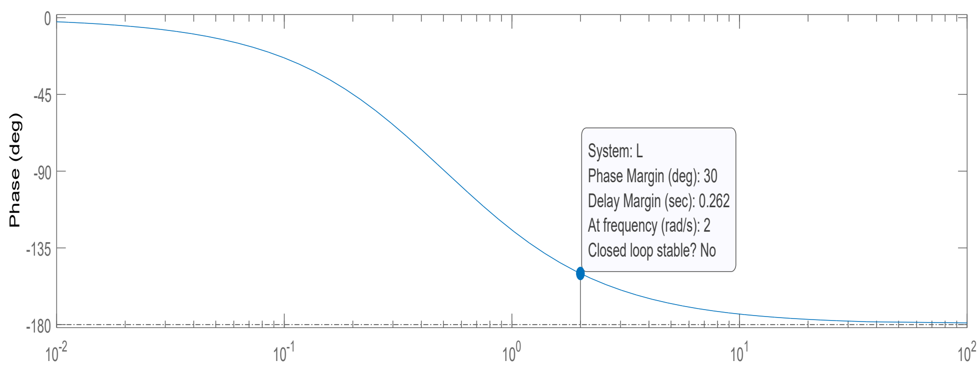

Figure 3: Bode plot of the $L(s)$ with the designed k_p and k_i.

Clearly, the system $L(s)$ has the desired phase margin at the desired frequency. An important observation is that the first equation in (59) is quadratic, which means that we

[4]See slide 56 for the first condition, and slide 61 for the definition of ϕ_m; both of Lecture 16. Or, see equations (9.5) and (9.6) of the book of Murray; the second in degrees.

will get two values for $k_p = \pm 0.3383$. We took the positive, because it adds the phase that is needed to reach the desired margin. If you take the negative value of k_p, you will get the reduced angle. In any case, I take it as correct in both cases.

Now, if you prefer to design the controller, in a closed-loop manner (feedback). Then, compute the closed-loop transfer function

$$\begin{aligned} G(s) &= \frac{L(s)}{1+L(s)} \\ &= \frac{12.3812(k_p s + k_i)}{s^3 + 0.7255s^2 + 12.3812k_p s + 12.3812k_i}, \end{aligned} \tag{61}$$

take $s = j\omega$, and evaluate $G(j\omega)$ as follows

$$G(j\omega) = \frac{12.3812(k_i + jk_p\omega)}{(12.3812k_i - 0.7255\omega^2) + j(12.3812k_p\omega - \omega^3)}. \tag{62}$$

From here, the following step is to apply the magnitude and phase conditions in (56), but for $G(j\omega)$, and find the gains k_p and k_i.

5. **Exercise 5 (10pts).** Consider the linearized model of Exercise 1.(d) with the parameter in Table 1.

   (a) (4pts) Compute the exponential matrix of A.

   SOLUTION:

   The exponential matrix is computed using the Cayley-Hamilton method (see Slide 12 of Lecture 10, and Tutorial 6) with the formula

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \cdots + \alpha(t)A^{n-1} = \sum_{k=0}^{n-1} \alpha_k(t)A^k. \tag{63}$$

   and

$$e^{s_i t} = \sum_{k=0}^{n-1} \alpha_k(t)s_i^k. \tag{64}$$

   where λ_i is the i-th eigenvalue of A. In this specific case, the system matrix A is of 2×2 dimension. Hence, $n = 2$ and equations (63) and (64), respectively, become

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A. \tag{65}$$

   and

$$\begin{aligned} e^{s_1 t} &= \alpha_0(t) + \alpha_1(t)s_1 \\ e^{s_2 t} &= \alpha_0(t) + \alpha_1(t)s_2. \end{aligned} \tag{66}$$

   From (33), the eigenvalues of matrix A are $s_1 = 0$ and $s_2 = -0. - 7255$. Substitution in (66), and solving for $\alpha_1(t)$ and $\alpha_2(t)$ yields

$$\alpha_0(t) = 1, \qquad \text{and} \qquad \alpha_1(t) = 1.3783(1 - e^{-0.7255t}). \tag{67}$$

   Finally, after substitution in (65), we get the exponential matrix

$$e^{At} = \begin{bmatrix} 1 & 1.3783(1 - e^{-0.7255t}) \\ 0 & 1 - 0.9999(1 - e^{-0.7255t}) \end{bmatrix} = \begin{bmatrix} 1 & 1.3783(1 - e^{-0.7255t}) \\ 0 & e^{-0.7255t} \end{bmatrix}. \tag{68}$$

(b) (3pts) Consider $u = -K_p x_1 - K_i \int_0^t x_1 d\tau + K_d x_2$. Write the closed-loop system.

SOLUTION:

For sake of completeness, let us rewrite the leariized system of Exercise 1 (d),

$$\delta \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2m} C\rho A_b \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ A_f \sqrt{\frac{2\rho g}{m A_b}} \end{bmatrix}}_{B} \delta u. \tag{69}$$

Notice that the controller is given in this item is for the state x of the nonlinear system, and not for the state δx for the linearization. Nevertheless, u is a linear control law since it is written as the sum of linear operations. Thus, around the operation point $(\overline{x}, \overline{u})$, the state x converges to δ, and so does the controller, i.e.,

$$u \approxeq \delta u = -K_p \delta x_1 - K_i \int_0^t \delta x_1 d\tau + K_d \delta x_2. \tag{70}$$

Now, because the integral operator is acting on the state δx_1, it defines a new state, which can be called $\delta x_3 = \int_0^t \delta x_1 d\tau$. By the fundamental theorem of calculus, we have $\delta \dot{x}_3 = \delta x_1$. It follows that system (69) in closed-loop with the controller (70) is given by

$$\begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_2 \\ \delta \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k_p & -0.7255 + k_d & k_i \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}. \tag{71}$$

(c) (3pts) What can you say if the degree of the numerator of a transfer function is bigger than the degree of the denominator?

SOLUTION:

It means that the transfer function is improper. That is, there are more zeros than poles. This is also related to lack of causality, in the signals and systems sense.